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Mapping the Climate

IRVING I. GRINGORTEN

19 January 1981

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FOR THE COMMANDER

John Howard

Chief Scientist

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For suitable graphical presentation of the climate, it is required that the mesoscale climate of a region, as small as 2 or 3 km², or as large as [illegible] 000 km², be described in sufficient detail to distinguish notable differences [illegible] the area. To this end, the construction of azimuthal equal-area [illegible] developed to achieve a high degree of conformality, as well as [illegible] the relative sizes of terrestrial areas. Polar equal-area world-[illegible] have been developed also with the north pole at the center of the maps. [illegible]

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20. Abstract (Continued)

the earth's surface is completely represented in a square. In another map the shape permits the optimum use of normal textbook or notebook pages.

Preface

A need for atlas-type climatology has prompted a re-examination of the map projections that have been used for hemispheric and world-wide climatic presentations. At the same time there has been a need for mesoscale regional studies of climatology, which requirement, for this report, has been viewed as a necessity for improved geographic presentations. Both types have been met with the development of equal-area mapping.

In this effort we have been assisted by the principal investigator, Mr. Jack Mettauer, under a Regis College contract, with computer solutions and plotting of regional equal-area maps.

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Mapping the Climate

1. INTRODUCTION

The main thrust of this paper is to detail a graphical means of presenting geographic information on climate for expeditious use. Climate, as defined by Landsberg[1] is the "collective state of the earth's atmosphere for a given place within a specified interval of time." The "place," as Landsberg explains, may be as small as an ant hill, as large as a continent, or in fact the whole earth.

This paper is conceived as a step toward the development of mesoscale climatology, or the study of the climate's temporal and spatial variability, within areas from a few square kilometers to several thousand square kilometers.

Studies of mesoscale climatology are numerous and varied partly because they need much adaptation to time and place. In the opinion of this writer, they do not permit practical generalizations for routine use. As a consequence, each region small or large, requires an individual description of its climatic characteristics. Yet for operational and planning purposes, it is important that the climate be presented in sufficient detail to distinguish important spatial differences, such as the increase in the frequency of cloud cover from the coastline toward the foothills of a west coastal region. An example of the effects of a large body of water, and

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(Received for publication 16 January 1981)

1. Landsberg, Helmut (1958) Physical Climatology, Gray Printing Co., Inc., DuBois, Pennsylvania, 446 pp.

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the distance of the station therefrom, can be perceived on diurnal wind variations east of Lake Michigan.[2]

The practice of presenting climatic information in map form is in line with recent developments in computer graphics. A compilation of papers, consisting "of ideas, practical methods and procedures, and illustrated applications," has been published recently by the Harvard Library of Computer Graphics.[3] Among these papers, one by Box[4] describes several ecological studies, which are facilitated by a system of "synagraphic" mapping, anachronized as SYMAP. Among the environmental elements presented graphically are climatic factors including temperature, precipitation, evaporation and solar radiation. Box describes the use of a new projection by Robinson[5] which represents a compromise between equal areas and correct shapes.

Programmed procedures for the computer-plotting of boundaries and climatic information onto a map have been described by others, with systems generally given code names. Recently a system, called EASYMAP[6] has been developed to depict statistical information by geographic areas, using shading, technically called choropleth. Present users include the Bureau of the Census and the United States Dept. of Commerce. While more attractive systems may be available for such mapping, aesthetics is sacrificed for economy and availability. The boundaries of geographic entities such as states or countries are plotted by the coordinates of many points, before the climatic information is plotted. Data sets, for this purpose, for the automatic plotting of shorelines, rivers, lakes, canals, political boundaries are now available.[7]

Samples of mesoscale mapping appear often in the meteorological literature. In Swan and Lee,[8] mapping is done in a rectangular area, including San Francisco,

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2. Weber, M.R. (1978) Average Diurnal Wind Variation in Southwestern Lower Michigan, J. Appl. Meteorol. 17:1182-1189.

3. Harvard University (1980) The Harvard Library of Computer Graphics Mapping Collection, Vol. 1–11, Center for Management Research, 850 Boylston St., Chestnut Hill, Massachusetts 02167.

4. Box, Elgene (1980) Use of Synagraphic Computer Mapping in Geoecology, The Harvard Library of Computer Graphics, 1979 Mapping Collection, Vol. 5, pp 11-27.

5. Robinson, A.H. (1974) A New Map Projection: Its Development and Characteristics, International Yearbook of Cartography 14:145-55.

6. Iskow, L.I., and Spoeri, R.K. (1979) EASYMAP: A Machine-Independent Line Printer Statistical Mapping System, The American Statistician 33:223-224.

7. Herman, Alan (1980) Computer Mapping Applications at the Atlantic Oceanographic and Meteorological Laboratories, Harvard Library of Computer Graphics, 1980 Mapping Collection, Vol. 8, pp 165-181.

8. Swan, P.R., and Lee, I.Y. (1980) Meteorological and Air Pollution Modeling for an Urban Airport, J. Appl. Meteorol. 19:534-544.

111 km north-south and 88 km east-west. In this small area are measurements made at some 13 sites, resulting in "interesting" mesoscale patterns of winds and air pollution. In addition, macroscale computer mapping has been illustrated with upper-air data in the southern hemisphere.[9]

In mapping we must decide on the nature of the statistical entries, in order to provide graphically information on the climate. The most obvious entry is a number properly located on the map. However, the purpose, the providing of information, may be served better by isopleths or isolines of equal values of the climatic element or parameter.

There are depictions of statistical information by shading (choropleth) or by colors. An example of shading can be found in the Climatic Atlas of the United States,[10] published by the United States Department of Commerce; for an example of the use of colors, the Climatic Atlas of Australia,[11] issued by the Australia Government Publishing Service may be cited. With the development of computer science and equipment, it is now possible to plot information automatically by use of shading or colors, for optimum presentation and use.[12]

There are advantages, but also obstacles in the use of graphics, especially for multidimensional data, such as ceiling and visibility; in fact, a satisfactory means of measuring the utility of a graphical scheme is still lacking. A fairly reliable element of personal judgment does exist in the selection of shading or gradation of color intensities. However, Wainer and Francolini[13] think that a legend of shading or colors is a "venial sin" if it must be memorized by the viewer in order to read the map.

2. ON THE CHOICE OF MAP PROJECTION

All maps inevitably distort the shape of a global land mass.[14] The earth's sphere cannot be mapped accurately onto a flat surface, although the approximation

9. Levanson, N., and Julian, P. R. (1980) Antarctic 150 mb Pressure Maps from TWERLE and Radiosondes (November 1975-March 1976), Monthly Weather Review **108**:520-526.

10. U.S. Dept. of Commerce (1968) Climatic Atlas of the United States, ESSA/EDS, U.S. Government Printing Office, Washington, D.C.

11. Bureau of Meteorology (1977) Climatic Atlas of Australia, Australia Government Publishing Service, Canberra.

12. Feinberg, S.E. (1979) Graphical Methods in Statistics, The American Statistician **33**:165-178.

13. Wainer, H., and Francolini, R.J. (1980) An Empirical Inquiry Concerning Human Understanding of Two-Variable Color Maps, The American Statistician **34**:81-93.

14. Raisz, E. (1962) Principles of Cartography, McGraw-Hill, New York, 315 pp.

can be good enough in a small area to make distortion almost imperceptible. Virtually, all global maps must show the earth's surface split, usually along one or more meridians. There are two aspects to avoid, distortion and splits, which themselves may dictate the choice of a map. But there are other factors to be dictated by the map's purpose.

Maps should have one or more features designated as follows: (1) azimuthal, (2) equidistant, (3) equal-area, and (4) conformal. To discuss these briefly:

1. In an azimuthal map, there is a central point (P_o), not necessarily representing the north pole, although it often does. From the central point, in all directions, points of the same earth distance from P_o will lie on a circle centered on P_o. Included among the azimuthal maps are the polar stereographic map and the polar Lambert azimuthal equal-area map.

2. An equidistant map is also azimuthal. The distance from the central point (P_o) of the map to another point (P) is an exact representation of the earth distance from P_o to P. This is a good map to have in a central airport, in order to estimate the true distance to another airport.

3. In an equal-area map, the sizes of all areas are accurately portrayed relative to other areas of the global surface. Such maps comprise a large family of projections, from which a selection can be made to meet other requirements such as economy of page space, freedom from distortion in selected locations, or absence of splits in important areas.

4. A conformal map, while conserving proper shapes of small regions such as a city, produces distortion of large areas such as the continents. But because of the freedom from distortion at single points, conformal projections are the maps of choice in navigation, supported by gnomonic charts for detailing the great circles or routes of shortest distance.

For statistical purposes, such as ours, the equal-area group of maps offers a distinct advantage. At the same time, there is an advantage when true land shapes are closely preserved, or when the distortion is lessened. Robinson[5] has presented a persuasive case for the general goal of creating new map projections with specific objectives, even though hundreds of projections now exist. Most cartographers have not engaged in such efforts because of other claims upon their time. However, with the computer-technology breakthrough, it is now much easier to generate new or novel base maps, and to use them readily by the same computer-plotting techniques.

The maps described in this paper are consistently equal-area. The earth is assumed to be absolutely spherical, with exactly 60 nm to one degree of latitude or great circle. To be used in the small areas of mesoscale studies, the shapes of azimuthal charts are correct and conformal at map center. From the center

outward, the distortion remains less than 1 percent even as far out as 700 nm from the center to be discussed later.

The earth's radius (R) has value:

$$R = (180 \times 60/\pi) \text{ nm} = 3,437.75 \text{ nm}$$

$$= 3,958.62 \text{ st mi}$$

$$= 6,366.71 \text{ km}$$

$$= 2.50818 \times 10^8 \text{ inches}$$

In an equal-area map, the linear scale will vary with orientation, but the geometric mean scale (s) will be constant throughout the map. Map sizes are given by the reciprocal of the map scale, as follows:

Map Size Nautical Miles to the inch	Scale s
1	1:72,960
10	1:729,600
100	1:7,296,000

or

Kilometer to the Centimeter	s
1	1:100,000
10	1:1,000,000
100	1:10,000,000

The symbol for latitude is ϕ, positive for latitude north, negative for latitude south. The symbol for longitude is λ, positive when measured east of the Greenwich meridian, negative when measured west of the Greenwich meridian. The spherical coordinates (ϕ, λ) are symbolized (ϕ_o, λ_o) for the central point of the azimuthal equal-area chart.

The maps described in this report are: (1) A regional map, centered on a selected point (ϕ_o, λ_o), produced to meet the requirements of a mesoscale study. It is always azimuthal equal-area. (2) World maps centered on the north pole are of two types: First, a square map that is polar azimuthal equal-area for the northern hemisphere, but fits the remaining southern hemisphere into the completed square,

for economy of space. Second, a world map centered on the north pole, split along four meridians so as to produce four quadrants, three of which display the continents unsplit, and with relatively little distortion except for Antarctica. The map fits neatly onto an (8-1/2 × 11) page of an atlas or other publication of conventional shape.

3. AZIMUTHAL EQUAL-AREA MAP CENTERED ON (ϕ_o, λ_o)

3.1 Construction

In all maps of this kind, there is a central point corresponding to the terrestrial coordinates (ϕ_o, λ_o), around which the map is drawn, preserving the equal-area feature. There is circular symmetry around the central point (Figure 1) at which there is zero distortion. Distortion gradually increases outward from the central point. When the projection is centered on the north pole, it is called Polar Lambert Azimuthal Equal-Area.

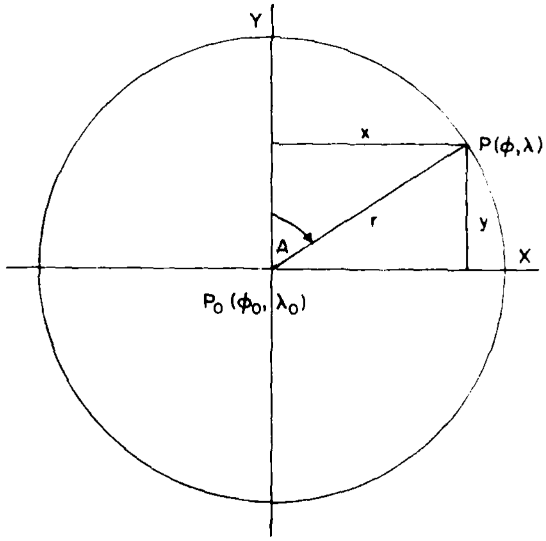

Figure 1. To Illustrate the Cartesian Coordinates of a Point (P) on an Azimuthal Map Centered on a Point (P_o) Whose Geographical Coordinates are (ϕ_o, λ_o). All points represented on the circle are geographically equidistant from $P_o(\phi_o, \lambda_o)$

On a spherical earth (Figure 2) the angular measurement, in radians, of any side of a spherical triangle (NP_oP) gives the earth distance along the great circle after multiplication by the earth's radius (R). The full length of the equator, like any great circle is $2\pi R$.

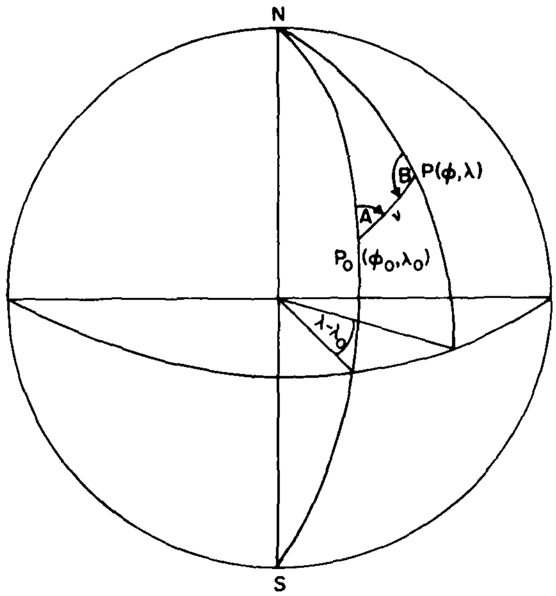

Figure 2. To Illustrate the Spherical Triangle NP_oP on a Sphere, in Which P_o Has Coordinates (ϕ_o, λ_o) and P Has Coordinates (ϕ, λ). The distance between P_o and P is measured by the great-circle arc (ν)

On an equal-area map of scale (s) the area of a global hemisphere ($2\pi R^2$) is mapped into the area of the circle of radius r_o, such that

$$2\pi R^2 s^2 = \pi r_o^2$$

which makes

$$r_o = \sqrt{2} \cdot (Rs) \tag{1}$$

Example: If the geometric mean scale (s) is 1:35,000,000 as it is in the 1976 working chart published by the Defense Mapping Agency (Figure 3) then r_0 is 10.135 inches; that is, the northern hemisphere is portrayed within a circle of radius 10.135 inches.

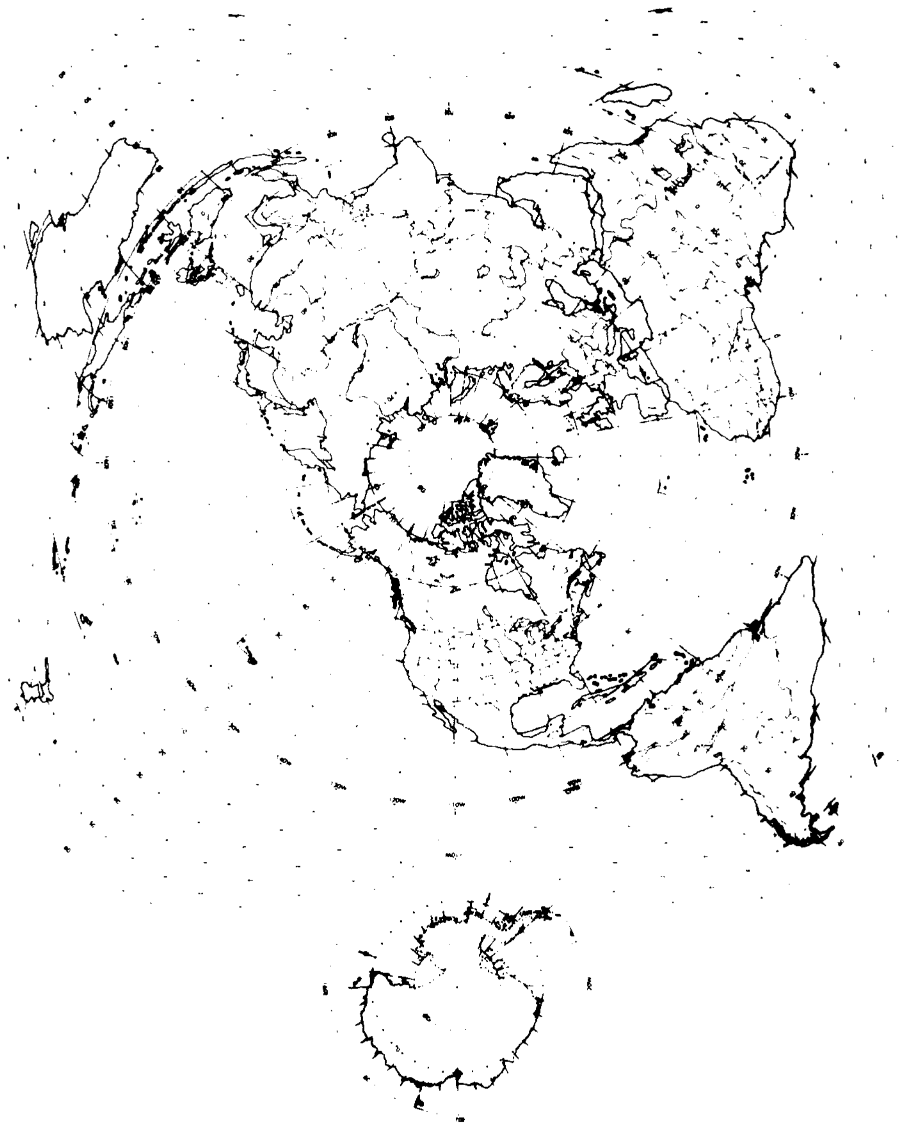

Figure 3. The Polar Equal-Area Map of the World. Prepared at request of the Air Force Geophysics Laboratory by the Defense Mapping Agency (PEA-3), June 1976

Consider the mapping of the global area surrounding the point (ϕ_o, λ_o). The small circle that subtends the conical angle ν at the center of the earth encloses the area $2\pi R^2(1 - \cos\nu)$. Hence the radius (r) of the corresponding circle on the map is such that

$$2\pi R^2(1 - \cos\nu) \cdot s^2 = \pi r^2$$

which makes

$$r = \sqrt{2(1 - \cos\nu)} \cdot (Rs) \tag{2}$$

On the earth (Figure 2), for the spherical triangle (NP_oP), the following equations hold:

$$\frac{\sin A}{\cos\phi} = \frac{\sin B}{\cos\phi_o} = \frac{\sin(\lambda - \lambda_o)}{\sin\nu} \tag{3}$$

$$\cos\nu = \sin\phi_o \sin\phi + \cos\phi_o \cos\phi \cos(\lambda - \lambda_o) \tag{4}$$

$$\sin\phi = \sin\phi_o \cos\nu + \cos\phi_o \sin\nu \cos A \tag{5}$$

On the map (Figure 1) the rectangular Cartesian coordinates (x, y) corresponding to (ϕ, λ) are given by

$$x = r \cdot \sin A \tag{6}$$

$$y = r \cdot \cos A$$

If $A > 90^o$, then $\cos A < 0$, to make y negative. For $A = 90^o$, Eq. (5) gives the corresponding ϕ-value as:

$$\sin\phi_s = \sin\phi_o \cos\nu_s$$

so that ϕ_s must have the same sign as ϕ_o for $\nu_s < 90^o$. Or both ϕ, ϕ_s are in the same hemisphere. Together with Eq. (3)

$$\cos\phi_s = \sqrt{\frac{1 - \sin^2\phi_o}{1 - \sin^2\phi_o \cdot \sin^2(\lambda - \lambda_o)}} \tag{7}$$

For $A > 90^o$, $\phi < \phi_s$ for positive or negative ϕ_s. Or, conversely, when $\phi < \phi_s$, then A is taken as equal to π minus the acute angle given by Eq. (3), making the y-component negative.

Example: Suppose we wish to draw the parallels and meridians surrounding Boston, Mass. (42°22'N, 71°2'W). Table 1 shows the results of computer computation for the coordinates (x, y) given the spherical coordinates (ϕ, λ). The computer was programmed (Appendix A) to solve for (x, y) by the above equations in the order: Eqs. (4), (3), (7), (2), (6).

Table 1. Map Coordinates Corresponding to the Given Point of Intersection of Meridians and Parallels Surrounding Boston, Massachusetts, as Computed, with $R = 2.508 \times 10^8$ inches, s = 1:614, 600, for ϕ_o:42°22'N;λ_o:71°2'W

Latitude	Longitude	x (inches)	y (inches)
42°00'N	70°30'w	2.82	-2.60
	71°00'W	0.18	-2.61
	71°30'W	-2.47	-2.60
42°30'N	70°30'W	2.80	0.95
	71°00'W	0.18	0.95
	71°31'W	-2.45	0.96

Table 2 shows the computed Cartesian coordinates for selected stations in the Boston area, which are plotted in Figure 4. Figure 4 retains the square grid by which all points were plotted. Each square corresponds to 243.4 km^2 of terrain. Another example of Cartesian coordinates (x, y), found to correspond to geographic coordinates (ϕ, λ) appears in Table 3 and plotted in Figure 5. It illustrates the effectiveness of the system in equatorial regions, around the international date line, in fact anywhere. However, to avoid computational embarrassments (such as zero divided by zero) the central point (ϕ_o, λ_o) was taken north of the equator 31 meters, corresponding to 1 second of angle. The plots in Figures 6, 7, 8, 9 show meridians and parallels that were plotted and drawn by computer automatically. The program was devised and written by the analyst, Jack Mettauer, under a contract with Regis College.

Table 2. The Cartesian Coordinates (x, y) of Stations Given by Latitude and Longitude on a Map of Scale s = 1:614, 600 Centered on Boston, Massachusetts ($42^{O}22'N, 71^{O}2'W$)

Station	ϕ	λ	x	y
Natick	$42^{O}17'N$	$71^{O}21'W$	-1.67	-0.59
Franklin	$42^{O}5'N$	$71^{O}24'W$	-1.94	-2.01
Lexington	$42^{O}27'N$	$71^{O}14'W$	-1.05	0.59
Needham	$42^{O}17'N$	$71^{O}14'W$	-1.05	-0.59
Brockton	$42^{O}4'N$	$71^{O}1'W$	0.09	-2.14
S. Weymouth	$42^{O}9'N$	$70^{O}56'W$	0.53	-1.54
Concord	$42^{O}28'N$	$71^{O}21'W$	-1.66	0.72

Figure 6 was drawn for Germany, centered on ($51^{O}N$, $10^{O}E$) with two locations plotted (B for Berlin, M for Munich). Figure 7 was drawn for Ecuador, centered on ($1^{O}30'S$, $78^{O}00'W$), with the locations of Lasso and Queto plotted by computer.

Figure 8 was drawn for Australia in the Southern hemisphere with central point at $25^{O}S$, $135^{O}30'E$. Figure 9 was drawn also for Australia, but the azimuthal map's center was chosen at $40^{O}N$, $100^{O}E$. Clearly this map, while still equal-area, is no longer conformal, even approximately.

3.2 Map Distortion

On the sphere (Figure 2) the incremental earth distance caused by an increase ($\delta\nu$ radians) in the arc (ν) is $R \cdot \delta\nu$. The incremental distance caused by an increase (δA) in the angle A (radians) is $R \cdot \sin\nu \cdot \delta A$. When the two are equal,

$$\delta A = \delta\nu/\sin\nu \tag{8}$$

On the map (Figure 1) the two corresponding increments will generally not be equal, and their ratio will be a measure of the distortion (e) thus:

$$1 + e = r \cdot \delta A/\delta r$$

which, together with Eq. (2), its differentiation and Eq. (8), gives

$$e = 2/(1 + \cos\nu) - 1 \tag{9}$$

Hence, at P_0, where $\nu = 0$, there is no distortion. But the distortion increases as ν increases. It will reach 1 percent when $\nu = 11.42^{\circ}$ which corresponds to an earth distance of 685.27 nm, or 789.1 st. miles or 1,269.12 km.

Table 3. Cartesian Coordinates (x, y) for the Azimuthal Equal-Area Map of the South Pacific, from 10°N to 10°S, 170°E and 170°W, Centered on (00°00'01"N, 180°00'00"W)

Name	Latitude	Longitude	x inches	y inches
Gilbert Island	00°00'	174°5'E	-1.29	0.00
Funafuti Atoll	9°00'S	179°00'E	-0.22	-1.97
Nukunono	9°12'S	171°55'W	1.75	-2.02
Canton Island	2°46'S	171°43'W	1.81	-0.61
Gardner	4°41'S	174°34'W	1.19	-1.03
Hull	4°30'S	172°14'W	1.69	-0.99
Beru	1°21'S	175°58'E	-0.88	-0.30
Tarawa	1°21'N	172°56'E	-1.55	0.30
Nanumea	5°39'S	176°06'E	-0.85	-1.24
Nui	7°16'S	177°10'E	-0.62	-1.59
Baker Island	00°13'N	176°28'W	0.77	0.05
	10°00'N	170°00'E	-2.16	2.19
		180°00'	0.00	2.19
		170°00'W	2.16	2.19
	0°00'	170°00'E	-2.19	0.00
		180°00'	0.00	0.00
		170°00'W	2.19	0.00
	10°00'S	170°00'E	-2.16	-2.19
		180°00'	0.00	-2.19
		170°00'W	2.16	-2.19

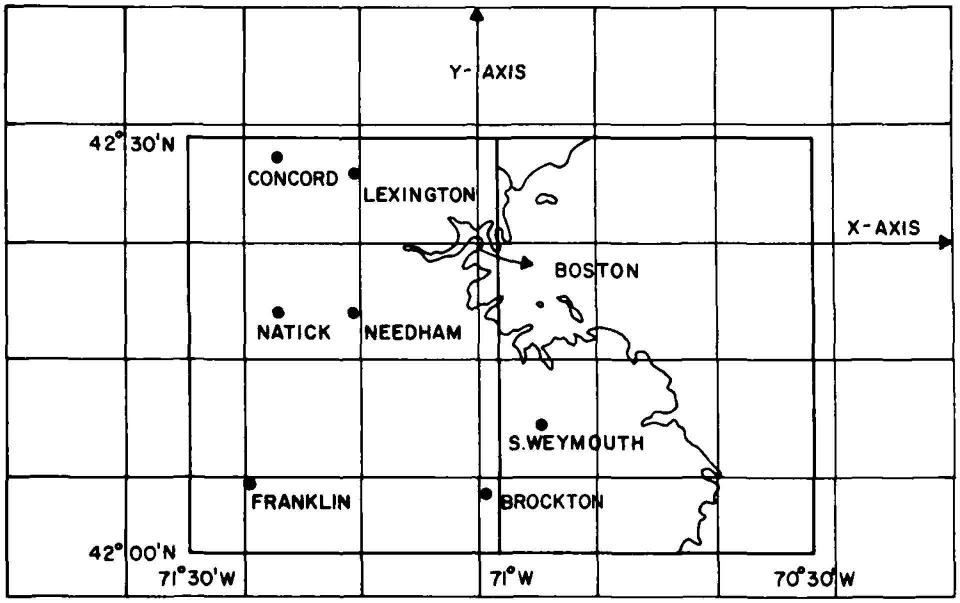

Figure 4. Azimuthal Equal-Area Projection Centered on Boston, Massachusetts. The map extends 55.6 km north-south and 82.2 km east-west. Each cell of the square grid corresponds to 243.4 km^2 of the earth's surface

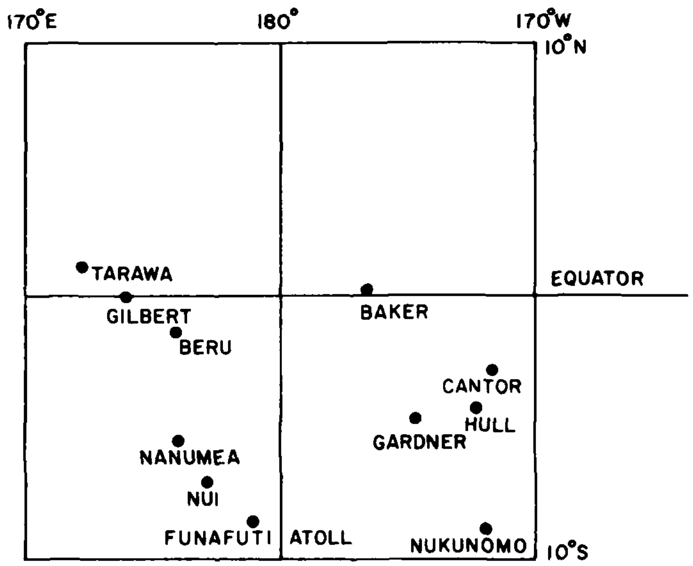

Figure 5. Azimuthal Equal-Area Map of the South Pacific, from 10°S to 10°N, 170°E to 170°W, Centered on the Intersection of the Equator and the International Dateline

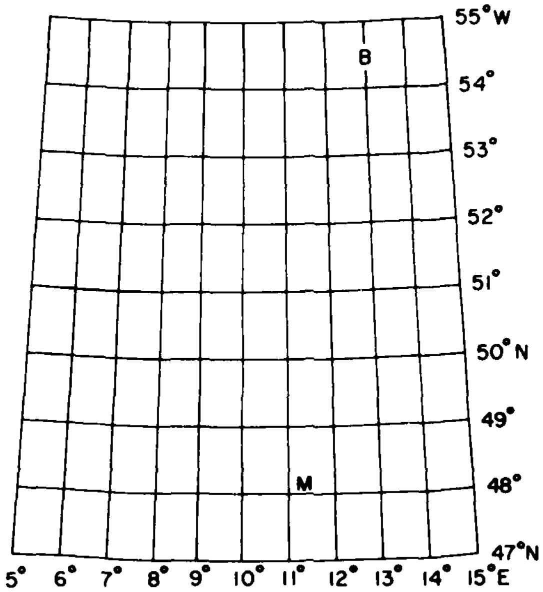

Figure 6. Meridians and Parallels for Germany

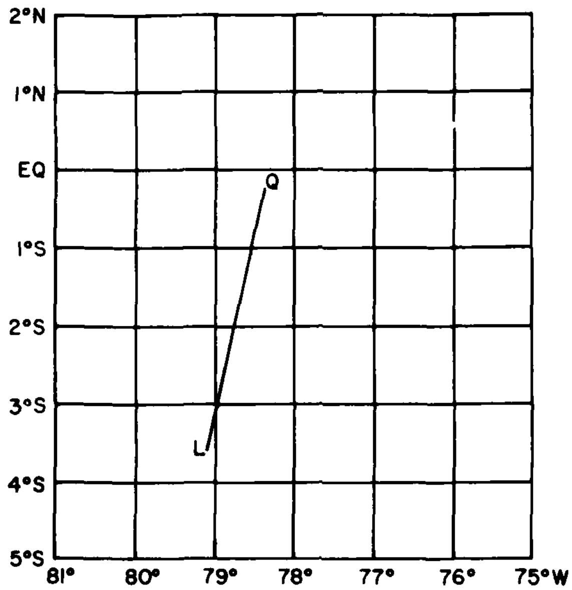

Figure 7. Meridians and Parallels for Ecuador. Point L is at Lasso, point Q at Queto

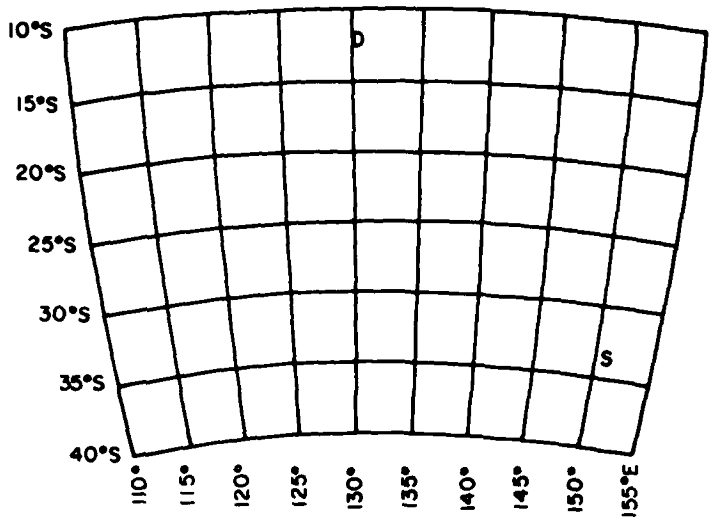

Figure 8. Meridians and Parallels for Australia on an Azimuthal Equal-Area Map Centered in Australia at 25°S, 132°30'E

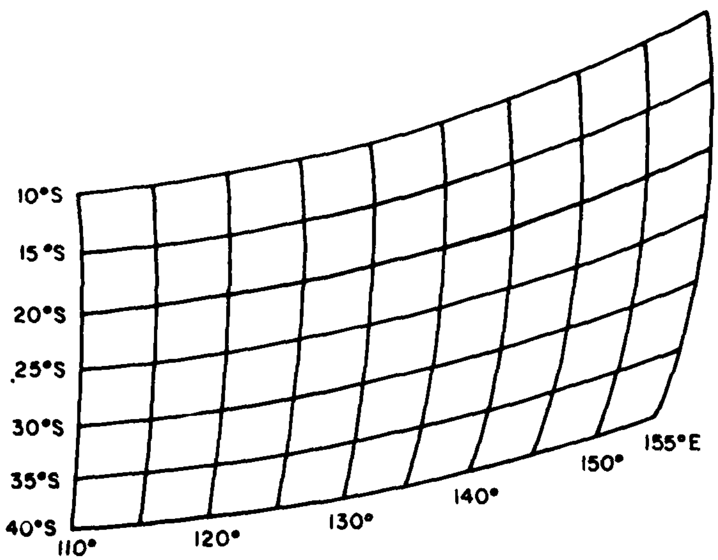

Figure 9. Meridians and Parallels for Australia on an Azimuthal Equal-Area Map Centered to the Northwest of Australia, at 40°N, 100°E

4. EQUAL-AREA MAPPING CENTERED ON THE NORTH POLE

4.1 Mapping North at Latitude ϕ_T

When the central point (ϕ_o, λ_o) is taken at the North Pole the projection becomes the well-known Polar Azimuthal Equal-Area Projection for the northern hemisphere. As in Eq. (4) the arc (r) becomes the co-latitude, so that, from Eq. (2) the map distance (r) from the north pole to a point (P) becomes:

$$r = \sqrt{2(1 - \sin \phi)} \cdot (Rs) \tag{10}$$

Since, at the north pole, λ_o does not have a unique value as defined so far, it needs to be redefined as a reference meridian from which others are measured. The Cartesian coordinates become:

$$x = r \cdot \cos (\lambda - \lambda_o) \tag{11}$$

$$y = r \cdot \sin (\lambda - \lambda_o)$$

4.2 Polar Equal-Area Mapping South of Latitude ϕ_T

Equations (11) are used in the construction of equal-area maps, suitable outward from the central point as far as the full northern hemisphere, as is done in the DMA map (Figure 3). For mapping beyond the hemisphere, or at some other desired distance from the central point, a new approach is adopted.

The construction, as described below, pertains primarily to the southern hemisphere of an equal-area map centered on the north pole. While it may be convenient to begin this mapping at the equator, it could be started at some latitude (ϕ_T) north of the equator. Such mapping will produce a top-of-the-world view of the continents (for example Figure 3 or Figure 10 or Figure 13). Except for Antarctica the continents need not be split, since the large land masses of the southern hemisphere can fall into three of four quadrants.

The plotting is done by Cartesian (x, y) coordinates (Figures 11 or 14) with the origin (0) representing the north pole and the X- and Y-axes representing two meridians at right-angles to each other. A key meridian is drawn as a diagonal straight line (OQ) where Q represents the south pole, one of four such points, each in one of the four quadrants.

If the angle

$$\beta = XOQ \tag{12}$$

is chosen greater than 45°, the map will be longer than it is wide. If $\beta = 45°$, then, as in Figure 3 or 10, the map will fit neatly into a square. If $\beta = 55°$, as in Figure 13, the map will fit neatly onto an 8-1/2 X 11 page.

In Figure 11 or 14, the meridian (λ) that makes an angle (γ) with the key meridian (OQ) is drawn straight from O to T, where T is on the circle representing the parallel (ϕ_T). The relation between λ and γ is

$$\gamma = \beta - \lambda + \lambda_o \tag{13}$$

where λ_o is the longitude of the X-axis. In the DMA map $\lambda_o = -20°$ for the meridian 20°W. Beyond T the meridian is represented by a curve (TQ), so that the area (OTQ) between the two meridians (OSQ, OTQ) on the map is proportionally equal to the corresponding area of the globe.

The total global area is $4\pi R^2$. Between two meridians subtending angle γ (radians) at either pole the area is $2\gamma R^2$. On the map the corresponding area is $2\gamma(Rs)^2$. But from the north pole southward to latitude (ϕ_T) the global area is $\gamma(1 - \sin\phi_T) \cdot R^2$. On the map the corresponding area is

$$\text{area (OTN)} = \gamma(1 - \sin\phi_T) \cdot (Rs)^2 \tag{14}$$

For the balance of this development assume

$$Rs = 1$$

and write

$$\text{area (OTN)} = \gamma(1 - \sin\phi_T) \tag{15}$$

For any point in the northern hemisphere with spherical coordinates (ϕ, λ) where $\phi > \phi_T$, the distance (r) from the north pole, on the map, is

$$r = \sqrt{2(1 - \sin\phi)} \tag{16}$$

The area, $A_\phi(\gamma)$, enclosed by latitude (ϕ) subtended by the apex angle (γ) is

$$A_\phi(\gamma) = \gamma(1 - \sin\phi) \tag{17}$$

The map distance, (OT), is given by

$$r_T = \sqrt{2(1 - \sin \phi_T)} \tag{18}$$

Set

$$OQ = k \tag{19}$$

The straight line VT is drawn tangent to the circle representing the latitude (ϕ_T), and QV is drawn perpendicular to VT, making

$$VT = k \sin \gamma \tag{20}$$

$$VQ = k \cos \gamma - r_T \tag{21}$$

Area NTQ = area OTQ - area OTN

$$= 2\gamma - \gamma(1 - \sin \phi_T)$$

$$= \gamma(1 + \sin \phi_T) \tag{22}$$

Also,

Area NTQ = Area OTS - area OTN + area VTQ - area VSQ

Hence,

$$\gamma(1 + \sin \phi_T) = 1/2\ r_T^2 \tan \gamma - \gamma(1 - \sin \phi_T) + A_c - 1/2\ VQ^2 \tan \gamma$$

where

$$A_c = \text{area VTQ} \tag{23}$$

After re-arrangement of terms,

$$A_c = 2\gamma + 1/2 \tan \gamma \cdot (VQ^2 - r_T^2) \tag{24}$$

At this stage a decision must be made on the shape of the curve TQ. Handily, if arbitrarily, it is made either elliptical or hyperbolic. The two alternatives are described, and used here.

4.2.1 HYPERBOLICALLY SHAPED MERIDIANS SOUTH OF THE EQUATOR – THE SQUARE MAP

In this section, the decision has been made that the shape of the meridians in the southern hemisphere will be hyperbolic. This is necessary if we are to achieve the goal of fitting the equal-area map into a square (Figure 10), because the outer boundary of the map, consisting of straight lines, is a limiting form of the hyperbola.

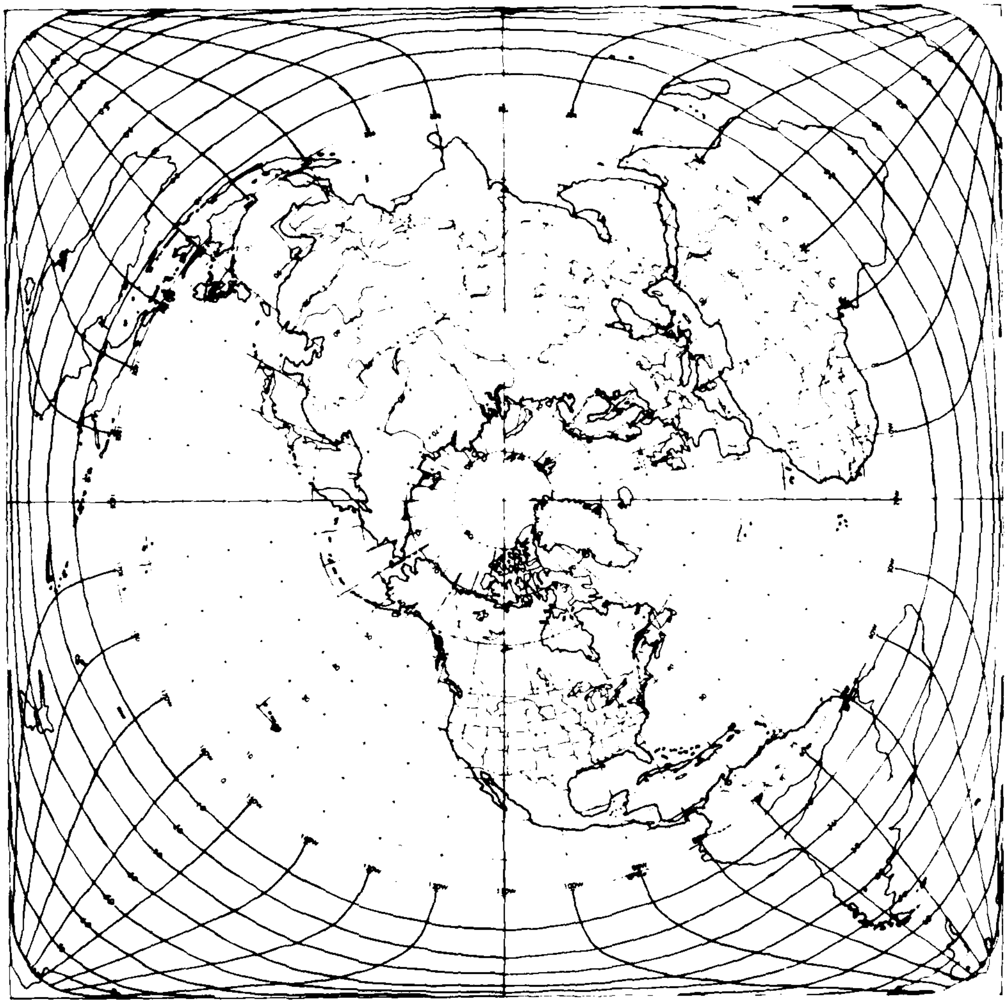

Figure 10. A Square Equal-Area Map of the World

Since the goal is [illegible] the [illegible] of a square map, the following [illegible] conditions are imposed:

$$\beta = 45^{\circ} \tag{2[illegible]}$$

$$k = \sqrt{2\pi}$$

The value of ϕ_T is set [illegible] 0, [illegible]

$$r_T = \sqrt{2}$$

The frame of reference (Figure 11) is transferred from the (X, Y)-plane to the (X', Y')-plane where the X'-axis is parallel to, but not identical with, the line VQ, and the Y'-axis is taken along the tangential line VT. Let the distance

$$O'T = b \tag{26}$$

For a hyperbolic shape the equation for the curve TQ can be written

$$x' = a\sqrt{y'^2 - b^2} + g(y' - b) \tag{27}$$

$$(y' \geq b)$$

where the parameters a, b, g need to be determined to conserve the equal-area characteristic of the map.

For Q, ($x' = VQ$, $y' = b + VT$), Eq. (27) gives

$$VQ = a\sqrt{(b + VT)^2 - b^2} + g \cdot VT \tag{28}$$

Differentiating Eq. (27), we obtain

$$1 = (ay'/\sqrt{y'^2 - b^2} + g) \cdot (dy'/dx') \tag{29}$$

Hence, at T ($x' = 0$, $y' = b$), $dy'/dx' = 0$

Or, the slope of the curve TQ makes it a smooth continuation of the straight line OT into the southern hemisphere, a desirable effect.

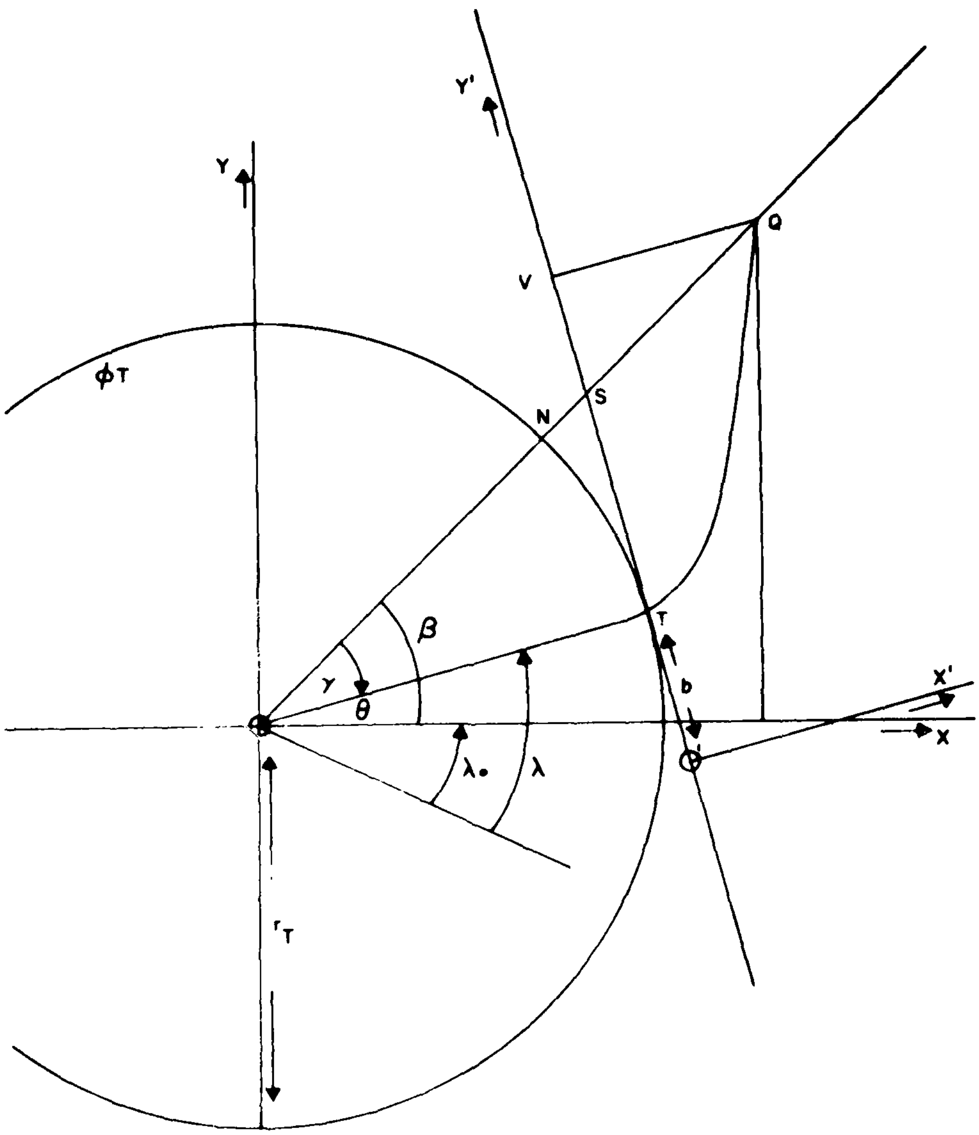

Figure 11. To Illustrate the Construction of a Square Equal-Area Map Centered on the North Pole

At Q make

$$\text{angle } TQV = 2\gamma' \tag{30}$$

leaving the actual value of γ' temporarily undetermined. At Q

$$dy'/dx' = \tan 2\gamma'$$

which, together with Eq. (29) gives

$$\tan 2\gamma' = h/(ac + gh) \tag{31}$$

where

$$c = (b + VT)/b \tag{32}$$

$$h = \sqrt{c^2 - 1}$$

which together with Eq. (28) give expressions for a, g in terms of b, thus:

$$a = h \cdot (VQ - VT \cdot \cos 2\gamma')/VT \tag{33}$$

$$g = \cot 2\gamma' - ac/h$$

The area VTQ given by Eq. (24) is also given by

$$A'_c = \int_{y=b}^{b+VT} x'\, dy'$$

which, with Eq. (27) becomes

$$A'_c = ab^2[c \cdot h - \ln(c + h)]/2 + g \cdot VT^2/2 \tag{35}$$

So far, the quantities b and γ' have been assumed known. But they must have values that will provide solutions, such that

$$A'_c = A_c \tag{36}$$

With some testing, γ' was chosen to be somewhat larger than γ, thus:

$$\gamma' = \gamma + \sqrt{\xi^2 - (22.5 - \theta)^2} - \sqrt{\xi^2 - (22.5)^2} \tag{37}$$

where $\xi = 50^{\circ}$ has been found to provide viable solutions. The value of b is found as accurately as desired by trial and error, to satisfy Eq. (36). With b known, the values of c, h, a and g are obtained from Eqs. (32), (33) and (34). (See Table 4.)

Table 4. Table of Parameter Values for the Square Equal-Area Map Centered on the North Pole, with Hyperbolically Shaped Meridians in Southern Hemisphere

θ	VQ	VT	a	b	g
0°	0.3582	1.7725	2.1365 5	1.677 -6	-2.1365 5
10°	0.6391	1.4377	3.6518	0.0934	-3.4374
20°	0.8576	1.0593	4.6118	0.0594	-4.0542
30°	1.0070	0.6488	6.8421	0.0341	-5.6404
40°	1.0829	0.2185	5.1095	0.0553	-1.3144
44.99°	1.0924	0.4375 -4	7.4164 2	0.7254 -4	9.5618 2

For each γ the curve TQ can be plotted by coordinates (x', y') using Eq. (27). Actually it is best to find the (X, Y)-coordinates for map plotting:

$$x = (x' + \sqrt{2}) \cos \theta - (y' - b) \sin \theta \tag{38}$$

$$y = (x' + \sqrt{2}) \sin \theta + (y' - b) \cos \theta \tag{39}$$

where

$$\theta = \beta - \gamma \tag{40}$$

The above solution yields pairs of coordinates (x, y) for points on a meridian (λ). But this has left the problem of finding (x, y) to correspond to a given latitude (ϕ) as well as longitude (λ).

Since the hyperbolic shape of the meridians in the southern hemisphere is assigned, the shapes of the parallels must follow as a consequence, subject to the constraint that the map is equal-area. Consider the point F (Figure 12) on the hyperbolic curve TQ representing the meridian of longitude (λ). The point F corresponds to the earth point (ϕ, λ) where ϕ is negative in the southern hemisphere. If an adjacent curve (T_1Q) is drawn such that the angle subtended by T_1T at O is small ($\delta\gamma$), then the small incremental area (F_1FQ) in the southern hemisphere must be made equal to the corresponding small increment in the northern hemisphere. That is,

$$\text{area } F_1FQ = \delta\gamma(1 - \sin(-\phi)) \tag{41}$$

But

$$\text{area } F_1FQ = \int_{y=y(\phi)}^{\sqrt{\pi}} (x - x_1) \cdot dy \doteq \sum_{i=1}^{n} (x - x_1)_i \cdot \delta y \tag{42}$$

where δy is a finite, but constant, increment that we can make as small as we please by dividing the Y-distance from Q to T into as many equal parts as we please. We make

$$\delta y = (\sqrt{\pi} - \sqrt{2} \cdot \sin \theta)/N \tag{43}$$

where N is large (1000 say). The summation of terms in Eq. (42) is such that the sum from Q to F is equal to the area given by Eq. (41). The number of such terms (n) increases as the latitude decreases in magnitude.

For the i^{th} term in Eq. (42)

$$y = \sqrt{\pi} - i\delta y \tag{44}$$

With x' given in terms of y' in Eq. (27), then Eq. (39) yields a value for y' as follows:

$$y' = (b + s) - \sqrt{s^2 - \{p^2/(\rho^2 - 1)\}} \tag{45}$$

where

$$p = (y - \sqrt{2} \sin \theta)/a \sin \theta \tag{46}$$

$$\rho = (g \sin\theta + \cos\theta)/a \sin\theta \tag{47}$$

$$s = (b + p \cdot \rho)/(\rho^2 - 1) \tag{48}$$

Finally x is given by Eq. (38) in terms of x' and y'.

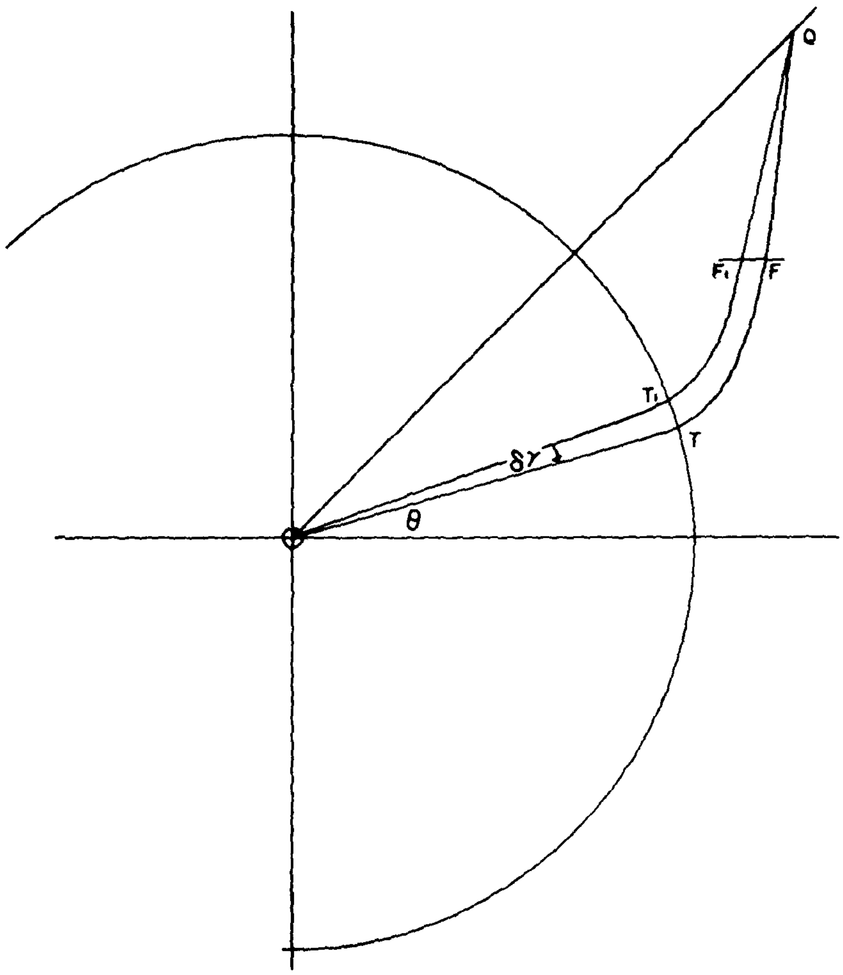

Figure 12. To Illustrate the Determination of the (X, Y)-coordinates on the Hyperbolic Curve (TQ) to Correspond to Given Spherical Coordinates (φ, λ)

The value of x is obtained on the curve QFT. Likewise the value of x_1 is obtained on the curve QF_1T_1. The summation of successive terms

$$(x - x_1)_i \cdot \delta y \qquad o < i < n$$

gives the approximate integration in Eq. (42) for the area FF_1Q. The summation increases as the magnitude of ϕ decreases.

An alternative development was used when θ was very small and for latitude close to the equator. Instead of the area F_1FQ the complementary area was used:

$$\text{Area } T_1TFF_1 = \delta\gamma \cdot \sin(-\phi)$$

for which, also:

$$\text{Area } T_1TFF_1 = \int_{x=\sqrt{2}\cos\theta}^{x(\phi)} (y_1 - y) \cdot dx = \sum_{i=1}^{n} (y_1 - y)_i \cdot \delta x \tag{49}$$

where

$$\delta x = (\sqrt{\pi} - \sqrt{2} \cdot \cos\theta)/N \tag{50}$$

For the i^{th} term

$$x = \sqrt{2}\cos\phi + i \cdot \delta x \tag{51}$$

With x thus given and x' given in terms of y' in Eq. (27) the value of y' is found from

$$y' = (b + s) - \sqrt{s^2 - \{p^2/(\rho^2 - 1)\}} \tag{52}$$

where

$$\rho = (\sin\theta - g \cdot \cos\theta)/(a \cdot \cos\theta) \tag{53}$$

$$p = (x - \sqrt{2} \cdot \cos\theta)/(a \cdot \cos\theta) \tag{54}$$

$$s = (b - p \cdot \rho)/(\rho^2 - 1) \tag{55}$$

With both x', y' thus found, the value for y is obtained from Eq. (39).

Likewise the value of y_1 is obtained. The summation of successive terms

$$(y_1 - y) \cdot \delta x \qquad o < i < n$$

gives the approximate integration in Eq. (49) for the area T_1TFF_1. The summation increases as the magnitude of ϕ increases.

The above procedure gives points in the lower half of the upper right-hand quadrant of the southern hemisphere, below line OQ (Figure 11 or 12). For the map above OQ the (X, Y)-coordinates are as follows:

$$\text{For } 45 < (\lambda - \lambda_o) \le 90^o \qquad \begin{aligned} x(\phi, \lambda) &= y(\phi,\ 90 - \lambda + \lambda_o) \\ y(\phi, \lambda) &= x(\phi,\ 90 - \lambda + \lambda_o) \end{aligned} \tag{56}$$

For points in the other three quadrants:

$$\begin{aligned}
&\text{For } 90 < (\lambda - \lambda_o) < 180 && x(\phi, \lambda) = -x(\phi,\ 180 - \lambda + \lambda_o) \\
& && y(\phi, \lambda) = y(\phi,\ 180 - \lambda + \lambda_o) \\
&\text{For } -180 < (\lambda - \lambda_o) < -90 && x(\phi, \lambda) = -x(\phi,\ 180 + \lambda - \lambda_o) \\
& && y(\phi, \lambda) = -y(\phi,\ 180 + \lambda - \lambda_o) \\
&\text{For } -90 < (\lambda - \lambda_o) < 0 && x(\phi, \lambda) = x(\phi, -\lambda + \lambda_o) \\
& && y(\phi, \lambda) = -y(\phi, -\lambda + \lambda_o)
\end{aligned} \tag{57}$$

Since the map is equal-area, it is possible to subdivide it into cells of equal area, that correspond to likewise equal areas on the globe, by latitudinal and longitudinal lines or curves. But, additionally since the map is square, it can be divided into equal areas by a square grid superimposed on the map.

For computer programming, the above procedure should be followed in steps (Appendix B). The Cartesian coordinates (x, y) corresponding to global points at $\phi = 0(-10^o)-90^o$, $\lambda = 0(10)40, 44.99$ are shown in Table 5.

4.2.2 ELLIPTICALLY SHAPED MERIDIANS SOUTH OF LATITUDE ϕ_T (Figure 13)

In Figure 14 the frame of reference is transferred from the (X, Y)-axes to the (X', Y')-axes. The X'-axis is taken along the line VT. The Y'-axis is parallel to, but not identical with, the line VQ. Let the distance

$$O'T = a \tag{58}$$

Table 5. The (x, y)-Coordinates, Corresponding to Points (ϕ, λ) on the Global Surface, on a Square Equal-Area Map, Which is Lambert Azimuthal in the Northern Hemisphere and With Hyperbolic Meridians in the Southern Hemisphere (Figure 10). Assume $\lambda_o = 0$

Long	Lat →	ϕ = 0 Equator	-10 10°S	-20 20°S	-30 30°S	-40	-50	-60	-70	-80	-90 South Pole
λ 0°	x	1.414	1.772	1.772	1.772	1.772	1.772	1.772	1.772	1.772	1.772
	y	0.000	-	-	-	-	-	-	-	-	1.772
10		1.393	1.506	1.596	1.661	1.698	1.722	1.738	1.751	1.763	1.772
		0.246	0.276	0.334	0.444	0.603	0.804	1.027	1.269	1.519	1.772
20		1.329	1.435	1.520	1.576	1.617	1.652	1.684	1.714	1.743	1.772
		0.484	0.535	0.620	0.744	0.894	1.058	1.231	1.409	1.589	1.772
30		1.225	1.322	1.397	1.458	1.512	1.565	1.616	1.668	1.720	1.772
		0.707	0.775	0.861	0.967	1.086	1.216	1.351	1.489	1.630	1.772
40		1.083	1.171	1.249	1.323	1.397	1.471	1.545	1.620	1.696	1.772
		0.909	0.990	1.075	1.165	1.258	1.356	1.456	1.560	1.665	1.772
44.99	x	1.000	1.085	1.169	1.253	1.338	1.425	1.511	1.599	1.686	1.772
	y	1.000	1.085	1.168	1.253	1.338	1.425	1.511	1.599	1.686	1.772

The parameters β, k, ϕ_T, and consequently r_T, have presently unassigned values. For an elliptical shape the equation for the curve TQ can be written:

$$y'/b = \sqrt{1 - (x'/a)^2} - g \cdot a \cdot (1 - x'/a) \tag{59}$$

where the parameters a, b, g need to be determined to conserve the equal area characteristic of the map.

For the point Q ($x' = a - VT$, $y' = VQ$) Eq. (59) gives

$$VQ/b = \sqrt{1 - \{(a - VT)/a\}^2} - g \cdot VT \tag{60}$$

Differentiating Eq. (59), we obtain

$$1/b \cdot dy'/dx' = g - x'/\{a^2 \sqrt{1 - (x'/a)^2}\} \tag{61}$$

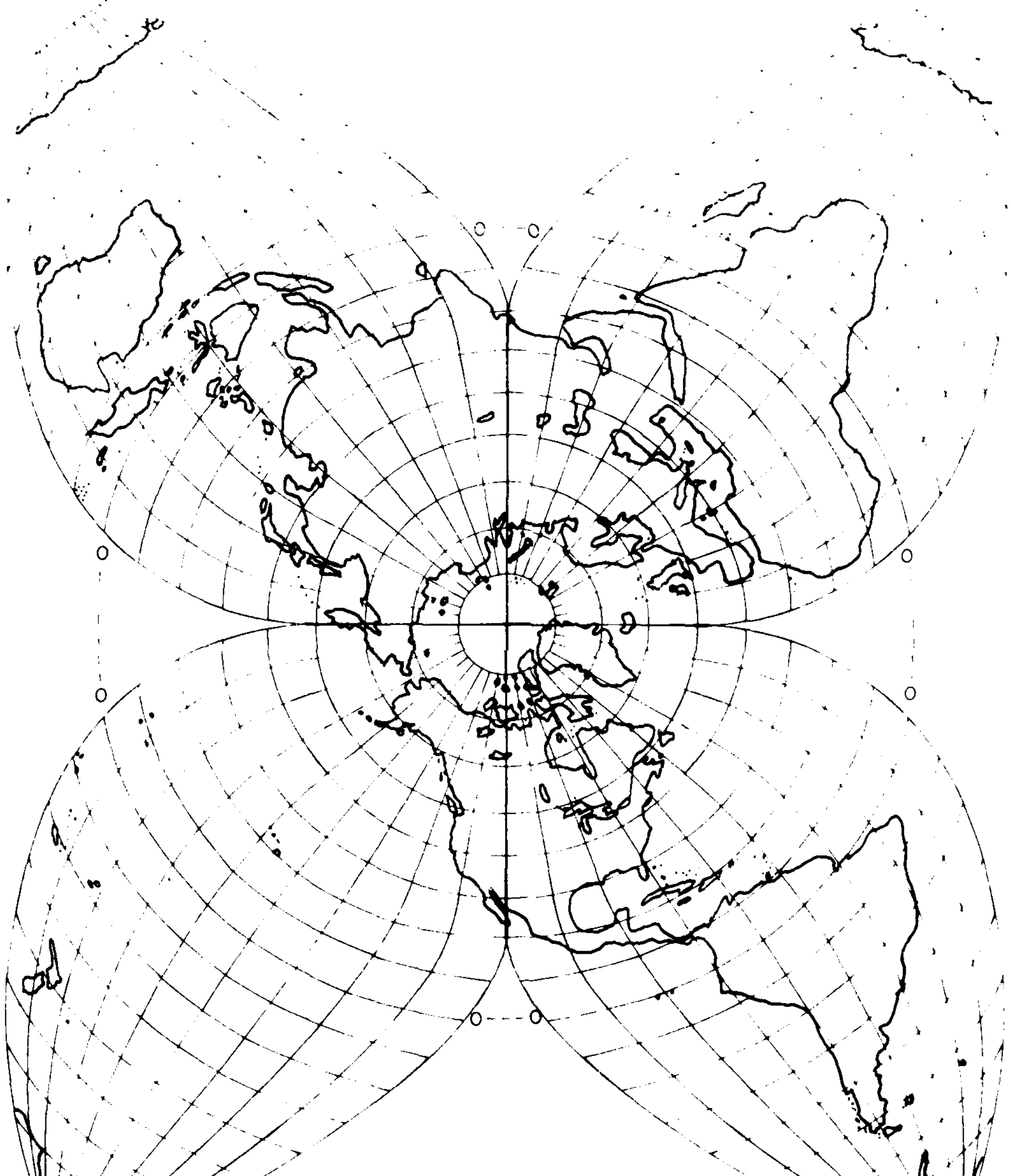

Figure 13. An Elongated Equal-Area Map of the World

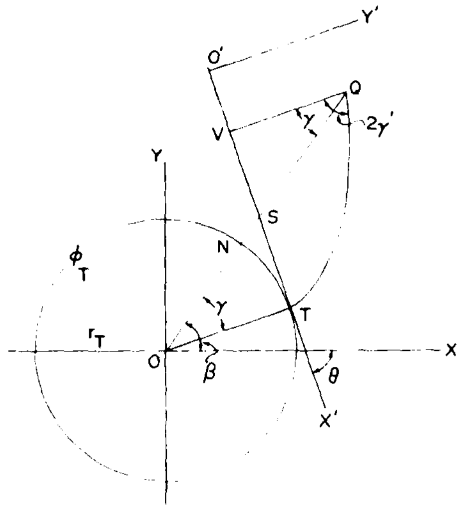

Figure 14. To Illustrate the Construction of the World Map With Elliptically Shaped Meridians in the Southern Hemisphere

Hence, $dy'/dx' \to -\infty$ as $x' \to a$ or, the slope of the curve TQ at T is perpendicular to VT, in order to make the curve TQ a smooth continuation of the line OT, a desirable effect.

At point Q, make

$$\angle TQV = 2\gamma' \tag{62}$$

where γ' is slightly larger than γ. Arbitrarily we make

$$\gamma' = \gamma \cdot \exp \{\gamma^{1/32} \cdot (\beta - \gamma)/3\} \tag{63}$$

where the angular measurements in the bracket are in radians, which makes

$$\gamma' = \gamma \text{ at } \gamma = 0$$

$$> \gamma \text{ at } 0 < \gamma < \beta$$

$$= \gamma \text{ at } \gamma = \beta \ .$$

At Q,

$$dy'/dx' = -\cot(2\gamma')$$

which, together with Eq. (61), gives

$$-\cot(2\gamma')/b = g - \{1 - (VT/a)\}/a\sqrt{1 - \{(a - VT)/a\}^2} \quad (64)$$

Equations (60) and (64) can be solved to obtain expressions for b, g in terms of a, thus:

$$b = h \cdot a[VQ - VT \cdot \cot(2\gamma')]/VT \quad (65)$$

$$g = h/VT - VQ/b \cdot VT \quad (66)$$

where

$$h = \sqrt{1 - \{(a - VT)/a\}^2} \quad (67)$$

The area VTQ given by Eq. (24) is also given by

$$A'_c = \int_{x'=a-VT}^{a} y' \cdot dx' \quad (68)$$

which, with Eq. (59), becomes

$$A'_c = (ab/2)[\pi/2 - (1 - VT/a) \cdot h - \sin^{-1}(1 - VT/a) - g \cdot a\,(VT/a)^2)] \quad (69)$$

Solution for a is obtained by making

$$A'_c = A_c \quad (70)$$

This can be done by trial and error, as accurately as desired. It has been done to 12 decimal places in approximately 40 iterations. When the value of a is put in final form, the values of b and g are given by Eqs. (65) and (66).

Once the values for a, b, g are obtained, for the given γ, the curves TQ can be plotted by coordinates (x', y') through Eq. (59). Actually it is best to find the (x, y)-coordinates for map plotting:

$$x = (y' + r_T) \sin \theta + (x' - a) \cos \theta \tag{71}$$

$$y = (y' + r_T) \cos \theta - (x' - a) \sin \theta \tag{72}$$

where

$$\theta = \pi/2 - \beta + \gamma \tag{73}$$

The foregoing solution yields pairs of Cartesian coordinates for plotting meridians. Since the elliptical shape of the meridians is assigned, the shapes of the parallels must follow as a consequence, subject to the constraint that the map is equal-area.

Consider the point F (Figure 15) on the elliptical curve TQ representing the meridian of longitude (λ). The point F corresponds to the earth point (ϕ, λ) where ϕ is negative in the southern hemisphere. If an adjacent curve (T_1Q) is drawn such that the selected angle subtended by T_1T at o is small ($\delta\gamma$), then the small incremental area (F_1FQ) in the southern hemisphere must be made equal to the corresponding small increment in the northern hemisphere as given by Eq. (17). That is,

$$\text{area } F_1FQ = \delta\gamma \, \{1 - \sin(-\phi)\} \tag{74}$$

But

$$\text{area } F_1FQ = \int_{y=y(\phi)}^{k \sin \beta} (x - x_1) \cdot dy = \sum_{i=1}^{n} (x - x_1)_i \cdot \delta y \tag{75}$$

where δy is a finite, but constant, increment, which we can make as small as we please by dividing the y-distance from Q to T into as many equal parts as we please; that is, we make

$$\delta y = \{(k \sin \beta - r_T \sin (\beta - \gamma))\}/N \tag{76}$$

where N can be made large, 1000 say. The summation of terms in Eq. (75) is made until the sum, from Q to F, is equal to the area given by Eq. (74). The number of such terms (n) increases as the latitude (ϕ) decreases in magnitude. The coordinates (x, y) at which Eq. (75) is satisfied become the locators of the point that has spherical coordinates ($-\phi, \lambda$).

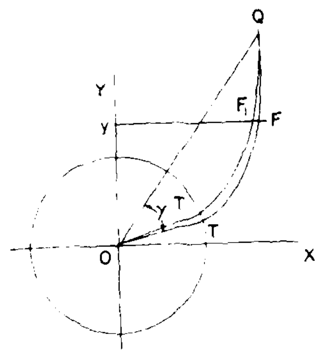

Figure 15. To Illustrate the Determination of the (X, Y)-coordinates on the Elliptical Curve (TQ) to Correspond to Spherical Coordinates (ϕ, λ)

For the i^{th} term in the summation (Eq. (75))

$$y_i = k \sin \beta - i \cdot \delta y \tag{77}$$

With this value of y

$$x' = (q + \sqrt{q^2 - p \cdot s})/p \tag{78}$$

where

$$p = c^2 + b^2 \cos^2 \theta / a^2$$

where

$$c = b\, g \cos \theta - \sin \theta$$

and where

$$s = (y + m)^2 - b^2 \cos^2 \theta$$

where

$$m = ac - r_T \cos \theta$$

and where

$$q = \cdot (y + m)$$

With y' given by Eq. (59), then

$$x = (y' + r_T) \sin \theta + (x' - a) \cos \theta \tag{79}$$

This value of x is obtained on the curve QFT. Likewise the value of x_1 is obtained on the curve QF_1T_1.

The summation of successive terms

$$(x - x_1)_i \cdot \delta y$$

gives the approximate integration for area F_1FQ. The summation increases as the magnitude of ϕ decreases. Sufficient accuracy is obtained when $\delta\gamma$ is sufficiently small, although it must not be too small for computer operability.

The above procedure was used to produce the coordinates in the upper right-hand quadrant of the map (Figure 13), at the intersection of meridians and parallels at 10° intervals (Table 6). The steps, for the computer are given in Appendix C.

Several key points and additions to the above analysis are needed. For the elongated map β (Figure 14) is greater than 45°. The lower part of the upper right-hand quadrant, below line OQ, is derivable by direct application of the above procedure. For the upper part, above line OQ, the computations should be done with the mirror image, in which β_m is less than 45° and giv·n by

$$\beta_m = 90^\circ - \beta \tag{80}$$

If the Cartesian coordinates so obtained are designated (x_m, y_m), then the primary coordinates are given by the equations:

$$x = x_m \cos 2\beta + y_m \sin 2\beta \tag{81}$$

$$y = y_m \sin 2\beta - y_m \cos 2\beta$$

Another key point concerns the latitude (ϕ_T) at which each meridian, beginning at the north pole, ceases to be represented by the straight line and begins its elliptical shape ending at the south pole. In the example (Figure 13) $\phi_T(X)$ is at 50°N along the X = axis and $\phi_T(Y)$ at 20°N along the Y-axis, and ϕ_T is a function of the longitude in the remainder of the quadrant, as follows:

Table 6. The Rectangular Coordinates (x, y) of the Intersection of Meridians and Parallels at 10° Intervals in the First Quadrant of the Elongated Polar Equal-Area Map

Longitude	20°W		10°W		0°		10°E		20°E	
γ	55°		45		35		25		15	
λ	-20		-10		0		10		20	
Latitude	x	y	x	y	x	y	x	y	x	y
N-Pole	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
80°N	0.174	0.000	0.172	0.030	0.164	0.060	0.151	0.087	0.134	0.112
70	0.347	0.000	0.342	0.060	0.326	0.119	0.301	0.174	0.266	0.223
60	0.518	0.000	0.510	0.090	0.486	0.177	0.448	0.259	0.397	0.333
50	0.684	0.000	0.674	0.119	0.643	0.234	0.592	0.342	0.524	0.440
40	0.859	0.014	0.862	0.161	0.820	0.302	0.732	0.423	0.647	0.543
30	1.026	0.056	1.009	0.220	0.956	0.375	0.879	0.523	0.777	0.659
20	1.178	0.123	1.149	0.303	1.084	0.469	0.996	0.627	0.881	0.770
10°N	1.320	0.218	1.279	0.411	1.203	0.584	1.103	0.745	0.979	0.890
Equator	1.447	0.338	1.393	0.538	1.309	0.715	1.200	0.873	1.070	1.017
10°S	1.557	0.484	1.493	0.686	1.401	0.860	1.286	1.012	1.154	1.149
20	1.648	0.653	1.577	0.855	1.479	1.018	1.363	1.159	1.233	1.287
30	1.719	0.843	1.641	1.037	1.543	1.188	1.431	1.315	1.308	1.430
40	1.767	1.054	1.687	1.231	1.592	1.363	1.487	1.474	1.375	1.574
50	1.791	1.284	1.713	1.437	1.627	1.548	1.535	1.639	1.437	1.720
60	1.790	1.527	1.720	1.652	1.647	1.738	1.571	1.805	1.493	1.870
70	1.762	1.782	1.707	1.870	1.653	1.930	1.598	1.976	1.542	2.018
80°S	1.707	2.046	1.674	2.094	1.644	2.123	1.615	2.146	1.586	1.167
S-Pole	1.622	2.317	1.622	2.317	1.622	2.317	1.622	2.317	1.622	2.317

Table 6. The Rectangular Coordinates (x, y) of the Intersection of Meridians and Parallels at 10° Intervals in the First Quadrant of the Elongated Polar Equal-Area Map (Continued)

Longitude	30°E		40°E		50°E		60°E		70°E	
γ	5		-5		-15		-25		-35	
λ	30		40		50		60		70	
Latitude	x	y	x	y	x	y	x	y	x	y
N-Pole	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
80°N	0.112	0.134	0.087	0.151	0.060	0.164	0.030	0.172	0.000	0.174
70	0.223	0.266	0.174	0.301	0.119	0.326	0.060	0.342	0.000	0.347
60	0.333	0.397	0.259	0.448	0.177	0.486	0.090	0.510	0.000	0.518
50	0.440	0.524	0.342	0.592	0.234	0.643	0.119	0.674	0.000	0.684
40	0.543	0.647	0.423	0.732	0.289	0.794	0.147	0.832	0.000	0.845
30	0.653	0.780	0.509	0.881	0.342	0.940	0.174	0.984	0.000	1.000
20	0.744	0.897	0.587	1.006	0.407	1.090	0.204	1.132	0.000	1.147
10°N	0.834	1.020	0.670	1.129	0.485	1.218	0.275	1.271	0.054	1.304
Equator	0.921	1.144	0.757	1.253	0.575	1.343	0.370	1.401	0.156	1.443
10°S	1.008	1.272	0.848	1.377	0.674	1.465	0.482	1.527	0.284	1.574
20	1.093	1.403	0.941	1.502	0.780	1.586	0.605	1.649	0.426	1.699
30	1.176	1.535	1.037	1.626	0.891	1.704	0.735	1.765	0.580	1.816
40	1.258	1.667	1.135	1.749	1.006	1.818	0.873	1.876	0.742	1.925
50	1.337	1.800	1.233	1.870	1.125	1.929	1.017	1.981	0.910	2.026
60	1.414	1.932	1.332	1.988	1.247	2.036	1.163	2.079	1.083	2.118
70	1.487	2.063	1.430	2.103	1.370	2.137	1.312	2.169	1.259	2.199
80°S	1.557	2.191	1.527	2.213	1.539	2.018	1.465	2.249	1.438	2.267
S-Pole	1.622	2.317	1.622	2.317	1.622	2.317	1.622	2.317	1.622	2.317

$$\phi_T = 50 - 1/3\ (\lambda - \lambda_o)$$

More generally

$$\phi_T = \phi_T(X) - (\lambda - \lambda_o) \cdot (\phi_T(X) - \phi_T(Y))/90 \qquad (82)$$

For the other three quadrants of the map, the (x, y)-coordinates can be found in terms of corresponding positions in the first quadrant by the transformations in Eq. (57).

Since the map is equal-area, it is possible to subdivide it into cells of equal area that correspond to equal areas on the globe as well (Figure 16). For the division of each hemisphere into M equal cells by latitudinal and longitudinal dividers, the latitude down to the nth ring will be given by

$$\phi_n = \sin^{-1} (1 - m_n/M) \qquad (83)$$

where m_n is the number of cells down to latitude ϕ_n. It is best to choose M divisible by four, to preserve the symmetry of the four quadrants.

There is a restriction in the use of the above procedure to construct an equal-area map. The solution is not always possible with the arbitrarily selected values. Figure 13 was obtained after several tests with varying values for the parameters: ϕ_T, β, k, γ', before the values were selected as $\beta = 55^o$, $k = 2\sqrt{2}$, $\phi_T(u) = 50^o$N, $\phi_T(\ell) = 20^o$N and γ' and ϕ_T given by Eqs. (63) and (82).

The Cartesian coordinates (x, y) corresponding to global points at $\phi = 90^o(-10)-90^o$, $\lambda = 0^o(10)90^o$ are shown in Table 6.

5. ILLUSTRATIONS

Maps of mesoscale characteristics are "everywhere," and world maps of specifics are commonplace. This report was written to advance the use of equal-area maps, both large-scale for mesoscale studies and small-scale for world-wide depiction. Admittedly, there have been few applications, so far, with which this report could be illustrated. Consequently there may be problems as yet unanticipated.

The world map issued by the Defense Mapping Agency (PEA-3) has been used satisfyingly. Figure 17 shows isopleths of average cloud cover, throughout the world, in January at local noontime. The centers of high frequency of cloud cover are marked with the plus sign (+), usually better than 8/10, and the centers of low frequency are marked with the minus sign (-), often lower than 2/10. Figure 18

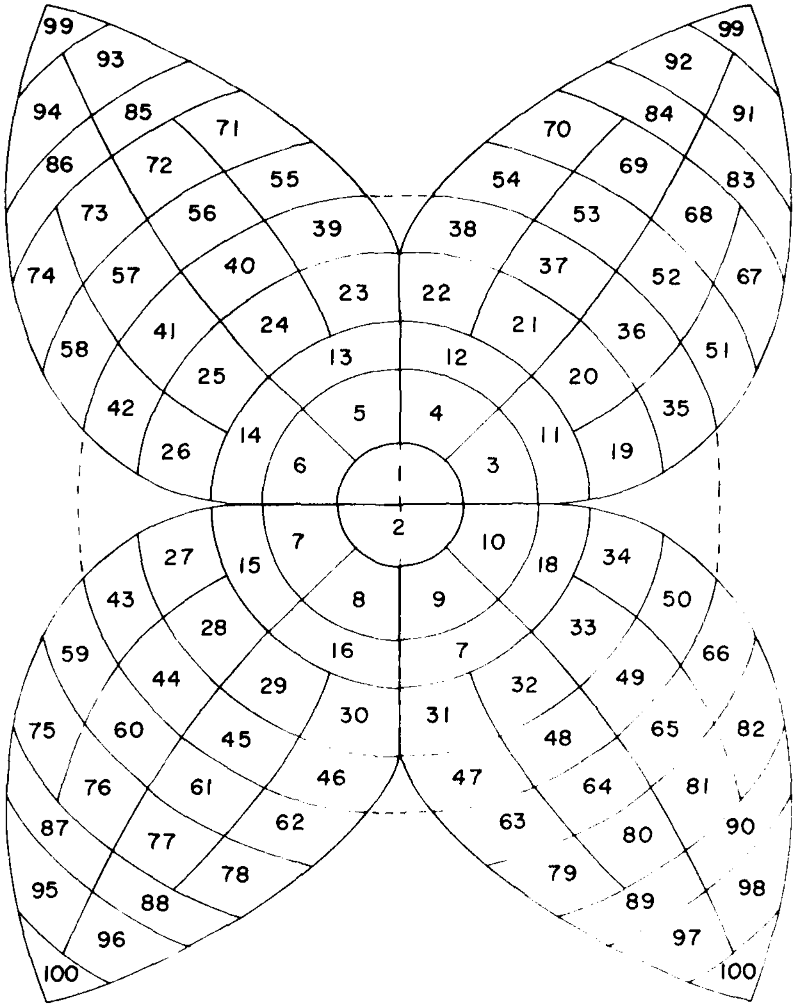

Figure 16. An Illustration of the Division of the Global Area into 100 Equal Cells, Each Bounded by Segments of Meridians and Parallels. (Cells 99, 100 are split, each between two quadrants)

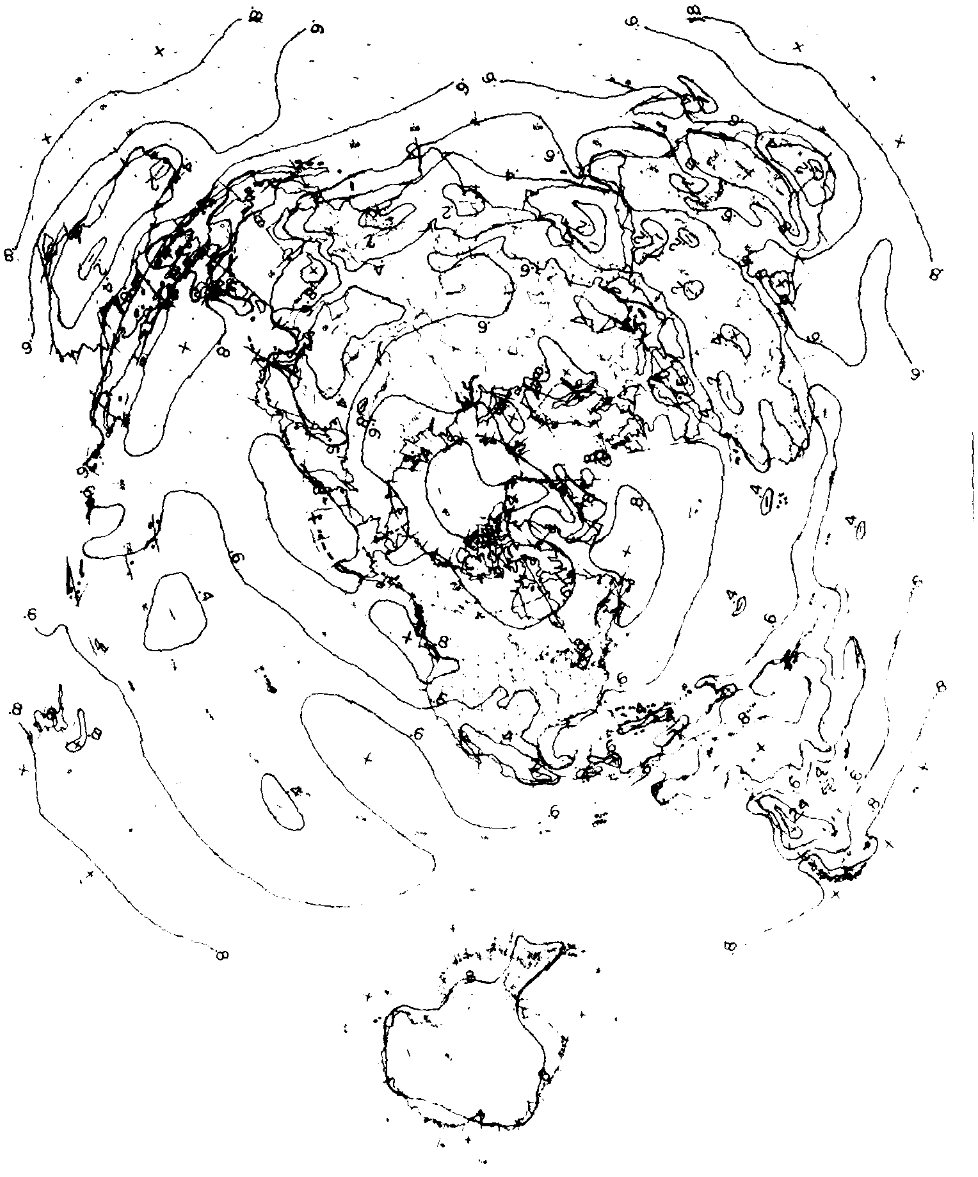

Figure 17. The PEA-3 Map with Isopleths of Average Cloud Cover, in January, Local Noontime

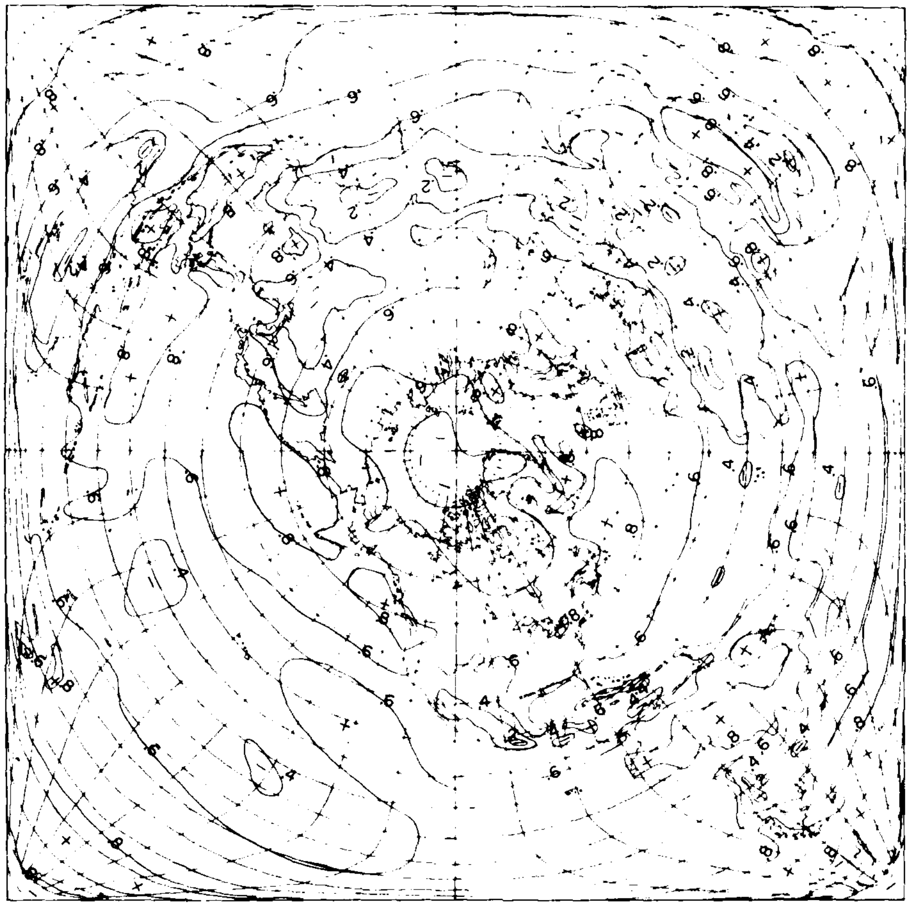

Figure 18. The Square Equal-Area Map Showing Average Cloud Cover, in January, Local Noontime

shows the same information on the square equal-area map, on which there actually are no breaks in the isopleths from one quadrant to the next, but at a cost. They are crowded in the higher latitudes of the southern hemisphere.

6. CONCLUDING REMARK

Much of this report has been devoted to the construction of the polar equal-area maps of the world, inasmuch as they are new "to the market." However, the main objective is to advance the use of azimuthal equal-area mapping, which can be computerized and expedited in all details.

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Appendix A

Program Steps for Plotting Cartesian Coordinates (x,y) Corresponding to Geographic Coordinates (ϕ,λ) on an Azimuthal Equal-Area Projection Centered on (ϕ_o,λ_o) Given the Map Scale (s)

Step 1. Enter the radius of the earth : $R = 2.508 \times 10^8$ inches

Enter the scale (s)

Step 2. Enter the coordinates of the central point (ϕ_o, λ_o)

$-90^o < \phi_o < 90^o \qquad -180^o \leq \lambda_o \leq 180^o$

Note: When $\phi_o = 90^o$, the map is the Polar Azimuthal Equal-Area Map, *which is treated separately.*

Step 3. Enter the coordinates of the station to be plotted (ϕ, λ).

Step 4. Find $\cos \nu = \sin \phi \sin \phi_o + \cos \phi \cos \phi_o \cos (\lambda - \lambda_o)$

Find ν, considered less than 90^o and positive.

Step 5. Find $\sin A = \{\cos \phi \cdot \sin (\lambda - \lambda_o)\}/\sin \nu$

Find A, which will be negative when $(\lambda - \lambda_o)$ is negative.

Note: Choose $|A|$ less than 90^o at this stage.

Step 6. Find $\cos \phi_s = \sqrt{(1 - \sin^2 \phi_o)/\{1 - \sin^2 \phi_o \cdot \sin^2 (\lambda - \lambda_o)\}}$

Find ϕ_s, which will be negative when ϕ_o is negative, as it is in the southern hemisphere.

Step 7. If $\phi < \phi_s$, replace A by $(180^\circ - A)$

Step 8. Find $r = \sqrt{2(1 - \cos \nu)} \cdot (Rs)$

Step 9. Find $x = r \sin A$

$y = r \cos A$

Print and/or plot (x, y).

Appendix B

Program Steps for Plotting Cartesian Coordinates (x,y) Corresponding to Geographical Coordinates (ϕ, λ) on the Square Equal-Area Map (Limited to the Octant: $0 \le x < \infty$, $0 \le y \le x$)

Assume that the earth's radius (R) times the map scale (s) are such that Rs = 1. Or, assume that all map distances are multiplied by (1/Rs).

Step 1: Enter λ_o, the meridian of reference (-20^o in the DMA-map)

Enter $\beta = 45^o$ (for a square map)

Enter $k = \sqrt{2\pi}$

Enter $\delta\lambda = 0.001^o$ (a suitable value)

Enter $\phi_T = 0$ (at the equator)

$r_T = \sqrt{2}$ (at the equator)

$N = 1000$ (or other suitable large number)

$\xi = 50$ (for a viable solution for γ')

Step 2. Enter latitude (ϕ) and longitude (λ) of the station.

Step 3. If $\phi < 0$, go to Step 4

For $\phi \ge 0$ (northern hemisphere)

find $r = \sqrt{2(1 - \sin\phi)}$

find $x = r\cos(\lambda - \lambda_o)$

find $y = r\sin(\lambda - \lambda_o)$ (END)

Step 4. For $\phi < 0$

If $\phi \geq -15^{\circ}$, go to Step 11.

For $\phi < -15^{\circ}$

find $A\phi(\delta\lambda) = \delta\lambda\{1 - \sin(-\phi)\}$

Enter $\Sigma (x - x_1)\,\delta y = 0$ (initial)

$y = \sqrt{\pi}$ (initial)

Find $\delta y = \{\sqrt{\pi} - r_T \cdot \sin(\lambda - \lambda_o)\}/N$

Step 5. Use SUBROUTINE A to find (and save) θ, b, a, g, ρ.

Step 6. Find $\lambda_1 = \lambda + \delta\lambda$

Use SUBROUTINE A to find (and save) θ_1, b_1, a_1, g_1, ρ_1

Step 7. With θ, b, a, g, ρ, y

Use SUBROUTINE B to find (and save) x

Step 8. With θ_1, b_1, a_1, g_1, ρ_1, y

Use SUBROUTINE B to find (and save) x_1

Step 9. Find $(x - x_1) \cdot \delta y$

and find new $\sum_{i=1}^{n} (x - x_1)_i \cdot \delta y$

If $\Sigma (x - x_1)_i \delta y < A_\phi(\delta y)$ go to Step 10.

For $\Sigma(x - x_1)_i \delta y \geq A_\phi(\delta y)$

read (x, y) (END)

Step 10. Decrease y by δy to give

$y(\text{new}) = y(\text{old}) - \delta y$

Go to Step 7.

Step 11. For $\phi \geq -15^{\circ}$

find $A_\phi(\delta\lambda) = \delta\lambda(\sin(-\phi))$

Enter $\Sigma(y_1 - y)\delta x = 0$ (initial)

$x = \sqrt{2} \cdot \cos(\lambda - \lambda_o)$ (initial)

Find $\delta x = \{\sqrt{\pi} - r_T \cdot \cos(\lambda - \lambda_o)\}/N$

Step 12. Use SUBROUTINE A to find (and save) θ, b, a, g, ρ

Step 13. Find $\lambda_1 \quad \lambda + \delta\lambda$

Use SUBROUTINE A to find (and save) θ_1, b_1, a_1, g_1, ρ_1

Step 14. With θ, b, a, g, ρ, x

Use SUBROUTINE B to find (and save) y

Step 15. With θ_1, b_1, a_1, g_1, ρ_1, x

Use SUBROUTINE B to find (and save) y_1

Step 16. Find $(y_1 - y) \cdot \delta x$

and find new $\Sigma(y_1 - y)_i \cdot \delta x$

If $\Sigma(y_1 - y)_i \cdot \delta x < A_\phi(\delta y)$ go to Step 17.

If $\Sigma(y_1 - y)_i \cdot \delta x \geq A_\phi(\delta y)$

read (x, y) (END)

Step 17. Increase x by δx to give

$x(\text{new}) = x(\text{old}) + \delta x$

Go to Step 14.

SUBROUTINE A.

(Using previous entries including λ, find θ, b, a, g, ρ.)

Step 1. Find $\theta = \lambda - \lambda_o$

$\gamma = \beta - \theta$

$VT = k \sin \gamma$

$VQ = k \cos \gamma - r_T$

$A_c = 2\gamma \text{ (rad)} + 1/2 \tan \gamma \, (VQ^2 - r_T^2)$

$\gamma'^o = \gamma^o + \sqrt{\xi^2 - (22.5 - \theta)^2} - \sqrt{\xi^2 - (22.5)^2}$

Step 2. Initialize $b(u) = 100$

$b(\ell) = 0$

$\nu = 0$

Step 3. Increase ν by 1.

Find $b = \{b(u) + b(\ell)\}/2$

$c = 1 + VT/b$

$h = \sqrt{c^2 - 1}$

$a = h[VQ - VT \cdot \cot 2\gamma'] / VT$

$g = \cot 2\gamma' - a \cdot c/h$

$A_c' = a \cdot b^2 [ch - \ln(c + h)]/2 + gVT^2/2$

Step 4. If $A_c' \geq A_c$

Make $b(\ell) = b$

Go to Step 5

If $A_c' < A_c$

Make $b(u) = b$

Step 5. If $\nu \geq 38$

Go to Step 6

For $\nu < 38$

Go to Step 3

Step 6. If $\phi < -15^o$

Find $\rho = (g + \cot \theta)/a$

(RETURN)

If $\phi \geq -15^o$

find $\rho = (\tan \theta - g)/a$

(RETURN)

SUBROUTINE B.

(Using θ, b, a, g, ρ, y or x, find x or y.)

Step 1. If $\phi \geq -15^o$

go to Step 2

For $\phi < -15^o$

find $p = \{y/\sin \theta - r_T\}/a$

$s = (b + p \cdot \rho)/(\rho^2 - 1)$

$y' = (b + s) + \sqrt{s^2 + \{p^2/(1 - \rho^2)\}}$

$x' = a\sqrt{y'^2 - b^2} + g(y' - b)$

$$x = (x' + r_T) \cos\theta - (y' - b) \sin\theta$$

(RETURN)

Step 2. For $\phi \geq -15^\circ$

$$p = \{x/\cos\theta - r_T\}/a$$

$$s = (b - p \cdot \rho)/(\rho^2 - 1)$$

$$y' = (b + s) + \sqrt{s^2 + \{p^2/(1 - \rho^2)\}}$$

$$x' = a\sqrt{y'^2 - b^2} + g(y' - b)$$

$$y = (x' + r_T) \sin\phi + (y' - b) \cos\phi$$

(RETURN)

Appendix C

Program Steps for Plotting Cartesian Coordinates (x,y) Corresponding to Geographical Coordinates (ϕ,λ) on a Polar Equal-Area Map with Elliptical Meridians South of Latitude ϕ_T (Limited to the Segment $0 \leq x < \infty$, $0 \leq y \leq x \tan \beta$)

Assume that the earth's radius ($R \approx 2.508 \times 10^8$ inches) multiplied by the map scale (s) gives Rs = 1. Or, assume that all map distances are multiplied by (1/RS).

Step 1. In this step the bracketed figures apply to Figure 13.

Enter λ_o, meridian of reference (-20°)

β, azimuthal angle of the key meridian (55°)

k, length of the key meridian, the diagonal straight line from north pole to south pole ($2\sqrt{2}$)

$\delta\lambda$, a very small increment of longitude (0.001°)

$\phi_T(X)$, the latitude at which the map is split along the X-axis (50°N)

$\phi_T(Y)$, the latitude at which the map is split along the Y-axis (20°N)

N, a large number (1000)

Step 2. Enter latitude (ϕ) and longitude (λ) of the station.

Step 3. Find $\phi_T = \phi_T(X) - (\lambda - \lambda_o) \cdot (\phi_T(X) - \phi_T(Y))/90$

Find $r_T = \sqrt{2(1 - \sin \phi_T)}$

Step 4. If $\phi < \phi_T$ go to Step 5

For $\phi \geq \phi_T$

Find $r = \sqrt{2(1 - \sin \phi)}$

$x = r \cos (\lambda - \lambda_o)$

$y = r \sin (\lambda - \lambda_o)$

(END)

Step 5. For $\phi < \phi_T$

Initialize $\Sigma(x - x_1)_i \cdot \delta y = 0$

$y = k \sin \beta$

Find $A_\phi(\delta\lambda) = \delta\lambda(1 - \sin(-\phi))$

$\gamma = \beta - (\lambda - \lambda_o)$

$\phi = 90^o + \gamma - \beta$

$\delta y = \{k \sin \beta - r_T \cdot \sin (\beta - \gamma)\}/N$

Step 6. Use SUBROUTINE A

to find (and save) θ, r_T, a, b, g, c, m, p

Step 7. Find $\lambda_1 = \lambda + \delta\lambda$

Use SUBROUTINE A

to find (and save) θ_1, r_{T1}, a_1, b_1, g_1, c_1, m_1, p_1

Step 8. With θ, r_T, a, b, g, c, m, p, y

Use SUBROUTINE B

to find (and save) x

Step 9. With θ_1, r_{T1}, a_1, b_1, g_1, c_1, m_1, p_1, y

Use SUBROUTINE B

to find (and save) x_1

Step 10. Find $(x - x_1)\delta y$

Find $\sum_{i=1}^{n} (x - x_1)_i \cdot \delta y$

If $\Sigma(x - x_1)\delta y < A_\phi(\delta y)$ go to Step 11

For $\Sigma(x - x_1)\delta y \geq A_\phi(\delta y)$

Read (x, y)

(END)

Step 11. For $\Sigma(x - x_1)\delta y < A_\phi(\delta y)$

Set y(new) = y(old) - δy

Go to Step 8.

SUBROUTINE A.

(Using previous entries, including λ, find θ, r_T, a, b, g, c, m, p.)

Step 1. Initialize a(u) = 100

$a(\ell) = 0$

$\nu = 0$

Find $\gamma = \beta - \lambda + \lambda_o$

$\phi = 90^o + \gamma - \beta$

$\phi_T = \phi_T(X) - (\lambda - \lambda_o) \cdot \{\phi_T(X) - \phi_T(Y)\}/90$

$r_T = \sqrt{2(1 - \sin \phi_T}$

$VT = k \sin \gamma$

$VQ = k \cos \gamma - r_T$

$A_c = 2\gamma \text{ (rad)} + 1/2 \tan \gamma \{VQ^2 - r_T^2\}$

$\gamma' = \gamma \cdot \exp \{1/3\ \gamma^{1/32} (\beta - \gamma)\}$

where the angles in the brackets are in radians.

Step 2. Increase ν by 1.

Find $a = \{a(u) + a(\ell)\}/2$

$h = \sqrt{1 - (1 - VT/a)}$

$b = h \cdot a[VQ - VT \cdot \cot 2\gamma'] / VT$

Find $g = h/VT - VQ/b \cdot VT$

Find $A_c' = (a \cdot b/2)[\pi/2 - (1 - VT/a) \cdot h - \sin^{-1}(1 - VT/a) - g \cdot a \cdot$

$(VT/a)^2)]$

Note: The $\sin^{-1}$ term must be in radians.

Step 3. If $A_c > A_c'$ set $a(\ell) = a$

If $A_c < A_c'$ set $a(u) = a$

Step 4. If $r < 38$ go to Step 2

For $r > 38$

Find $c = bg \cos\theta - \sin\theta$

$m = ac - r_T \cos\theta$

$p = c^2 + b^2 \cos^2\theta / a^2$

(RETURN)

SUBROUTINE B.

(Using previous entries, including y, find x.)

Step 1. Find $s = (y + m)^2 - b^2 \cos^2\theta$

$q = c(y + m)$

$x' = [q + \sqrt{q^2 - p \cdot s}]/p$

$y' = b\sqrt{1 - (x'/a)^2} - g \cdot a \cdot b(1 - x'/a)$

$x = (y' + r_T)\sin\theta + (x' - a)\cos\theta$

(RETURN)